

$J = 0, J = J_{max}$, and Quadrupole Pairing

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Abstract

We consider 2 neutrons and 2 protons in the $g_{9/2}$ shell. Wave functions and energy levels are obtained for various interactions. The wave functions for states with total angular momentum I greater or equal to 10 are not affected by what the pairing interaction ($J=0$ $T=1$) is. Other parts of the interaction are therefore of increased importance.

1 Introduction

In 1964, McCullen et al. [1] (MBZ) included detailed wavefunctions in the $f_{7/2}$ shell in their paper and an accompanied technical report. At that time the spectrum of the $J=0$ pairing Hamiltonian was well studied. For the lowest state of ^{44}Ti with this interaction the coefficients are:

$$C(0,0) = 0.8660, C(22) = .2152, C(44) = 0.2887, C(6,6) = 0.3469.$$

However MBZ obtained quite different coefficients with $C(0,0)$ smaller and $C(2,2)$ bigger. The wavefunction coefficients in an updated version of MBZ by Escuderos et al. [2] are:

$$C(0,0) = 0.7878, C(2,2) = 0.5615, C(4,4) = 0.2208, C(6,6) = 0.1234.$$

Whereas the ($J=0$ $T=1$) pairing interaction focuses on particles of one kind, with a system of both protons and neutrons one can have isospin $T=0$ interactions and the ones with $J=1^+$ and $J=J_{max}$ ($=7$) lie low.

For 2 protons and 2 neutrons in the $g_{9/2}$ shell, a remarkably similar wavefunction structure was obtained by the Swedish group [3]. The coefficients are:

$$C(0,0) = 0.76, C(2,2) = 0.57, C(4,4) = 0.24, C(6,6) = 0.13, \text{ and } C(8,8) = 0.14.$$

History repeats itself. They however developed the consequences of J_{max} pairing more sharply than was done in the past.

In a recent paper on maximum J pairing, Zamick and Escuderos [4] made a comparison of spectra and wavefunctions of various schematic interactions with those of a more realistic interaction. Specifically they considered 2 proton holes and 2 neutron holes relative to doubly magic ^{100}Sn , i.e. ^{96}Cd . These were single j shell calculations in the $g_{9/2}$ shell. In particular, in Table VI of [1], the authors compare overlaps of wavefunctions of two schematic interactions. We use the notation $E(J)$ for an interaction in which all two-body matrix elements are zero except the one with angular momentum J . In [1] we have compared the overlaps of wavefunctions arising from matrix diagonalizations with $E(9)$ and $E(0,9) = (E(0) + E(9))$ with wavefunctions arising from the realistic interaction CCGI [2] represented by 10 matrix elements $V(J) = \langle (jj)^I | V | (jj)^I \rangle$, $I=0,1,2,\dots,9$. These are respectively -2.3170, -1.4880, -0.6670, -0.4400, -0.1000, -0.2710, 0.0660, -0.4040, 0.2100, and -1.4020.

It should be mentioned that in re. [1] Table VI, to an excellent approximation, the $E(9)$ interaction for the wavefunctions of the lowest states were proportional to unitary $9j$ symbols $\psi = N \langle (jj)^9 (jj)^9 | (jj)^{J_p} (jj)^{J_n} \rangle^I$ with J_p and J_n both even and N approximately $1/\sqrt{2}$.

2 Overlaps

In this section we will consider overlaps $\langle\psi, \psi_{CCGI}\rangle$. Let us somewhat arbitrarily say that anything greater than 0.9 is a good overlap. When we use the $E(9)$ interaction, we get good overlaps for the lowest energy states with $I = 0, 2$, and 4. We get bad overlaps for $I = 6, 8$, and 10, the values being 0.6795, 0.2375, and 0.6860 respectively. For $I = 12, 14$, and 16, we again get good overlaps. When we use the $E(0, 9)$ interaction we get still good overlaps for $I = 1, 2, 3$, and 4 but in addition we get good overlaps for $I = 6$ and 8, the latter two being 0.9361 and 0.9858. That is to say equal attraction in The $J=0$ and $J=J_{max}$ channels cures the problem that is present when the only interaction is in the $J=9$ channel.

It should be noted however that when we switch from $E(9)$ to $E(0, 9)$ there is no change in the overlaps for $I = 10, 12, 14$, and 16. This is easy to understand. For these high angular momentum states there cannot be a pair of nucleons coupled to angular momentum zero, because the maximum angular momentum of the other two nucleons in the $g_{9/2}$ shell is 9.

This leaves $I = 10$ as a special state, still with a bad overlap of 0.6860. Clearly, since $J=0$ pairing is out of the picture, this state will be more sensitive to other parts of the interaction. We therefore consider $E(1, 9) = [E(1) + E(9)]$, $E(2, 9) = 1/2E(2) + E(9)$ as well as $E(1, 2, 9) = 1/3[E(1) + E(2) + E(9)]$. We take all the interactions to be attractive (negative). We get the following overlaps for $I = 10$:

Table 1: $I = 10$ Overlaps with Unitary $9j$ Coefficients

Interaction	Overlap
$E(9)$	0.6830
$E(2, 9)$	0.9004
$E(1, 9)$	0.9055
$E(1, 2, 9)$	0.9659

Note that $E(2, 9)$ and $E(1, 9)$, although quite different interactions, give almost the same overlaps. This means it is dangerous to determine the parameters of an interaction solely on the basis of overlaps. We get the best overlap by lowering all 3 matrix elements, $J=1, 2$, and 9. This agrees qualitatively with the CCGI interaction [2].

Concerning the highest angular momentum state, $I=16$, this is a unique state so it is not surprising that the overlap with any interaction is one. What about $I = 12$ and 14? We note that for these not only does $E(0)$ not enter but also $E(2)$ ($2+9=11$). The fact that $E(0)$ and $E(2)$ do not enter is clearly shown in Fig. 1 of the work of Qi [3]. Related works by the ‘Swedish group’ are here also cited [3,4,5,6] as well as the almost-Swedish group [7].

Seniority arguments have been presented for the 8^+ state by Fu, et al. [8,9] and the ‘Swedish group’[4]. They note this state is not well described as $[9\ 9]^8$, i.e. 2 pairs of neutrons and protons each coupled to spin 9, but rather as seniority 2 state [8 0] and [0 8]. Here in the first term we have 2 protons coupled to 8^+ and 2 neutrons to 0^+ , etc. For CCGI, the probability of these 2 configurations is 66%.

3 Overlaps of $E(9)$ with $U9j$

In the following tables, we present overlaps of wavefunctions for the $E(9)$ interaction and the column vectors of the $U9j$ interaction properly normalized. Here $U9j = \langle(jj)^9(jj)^{J_B} | (jj)^{J_P}(jj)^{J_N} \rangle^I$. We see that with $J_B = J_{max} = 9$, we get overlaps very close to one for the lowest states for all even I . For $I = 2$, there are very strong overlaps for the first two states with associations $J_B = 9$ and 7 respectively.

We give special attention to $I = 4$. Beyond the lowest $I = 4$ state there are two nearly degenerate states at 3.028 MeV and 3.072 MeV. The overlaps are 0.626255 and 0.786196 for the lowest state corresponding to $J_B = 7$ and $J_B = 5$, respectively, and -0.779731 and 0.617937 for the upper state. However, we can take linear combinations of the two eigenstates of the $E(9)$ interaction $\alpha|1\rangle + \beta|2\rangle$ and $-\beta|1\rangle + \alpha|2\rangle$ such that the

overlaps are very close to one with the first state associated with $J_B = 7$ and the second with $J_B = 5$. The coefficients α and β are obtained as follows:

Let $\vec{a} = |1\rangle$ be the first $E(9)$ eigenstate, $\vec{b} = |2\rangle$ be the second, and \vec{U} be the U9j column vector for a given value of J_B . We use the Lagrange Multipliers method to maximize the function $f(\alpha, \beta) = (\alpha\vec{a} + \beta\vec{b}) \cdot \vec{U}$ with the constraint $g(\alpha, \beta) = \alpha^2 + \beta^2 = 1$. This method gives the extrema of f with constraint g as solutions to $\nabla f = \lambda \nabla g$ for some real constant λ . Using this method, we find

$$\alpha^2 = \frac{m^2}{m^2 + 1}, m \equiv \frac{\vec{a} \cdot \vec{U}}{\vec{b} \cdot \vec{U}}$$

and $\beta^2 = 1 - \alpha^2$. Note that m is simply the ratio of the overlaps.

Note that we assign isospin quantum numbers to the states in the appendix. We can do this because the $E(9)$ interaction and indeed all the interactions considered here are charge independent.

It should be noted that with the $E(0)$ interaction, the $T = 0$ ground state cannot be 100% (J_p, J_n) = (0, 0) [2]. This is because the unique $T = 2$ states must have substantial amount of this configuration. Recent work by K. Neergaard shows that one can get improved $I = 0^+$ wavefunctions in ^{48}Cr , ^{88}Ru , and ^{92}Pd by admixing 75% (s, t) = (0, 0) and 25% (4, 0) where s is seniority and t reduced isospin for $Sp(2j + 1)$ [10].

4 Conclusions

In this simplified model, we find a priori the surprising behavior that overlaps of the $E(9)$ interaction with realistic interactions exceed the 0.9 limit for low angular momenta ($I = 0, 2, 4$), are well below the 0.9 limit for intermediate I (6, 8, 10), but again exceed this limit for large I (12, 14 and 16). Introducing $E(0, 9)$ cures the problem for $I = 6$ and 8 but not for $I = 10$. For $I = 10$, the details of the $J=2$ matrix element become super important (quadrupole pairing), but $J=2$ does not affect $I = 12, 14$, and 16.

It should be cautioned that overlaps can be deceptive. For example, for $I = 0$, the overlap with $E(9)$ is 94.5% [1], but the spectrum is such that the $J=16^+$ state is the ground state. It is 1.06 MeV below the lowest $I = 0^+$ state. With $E(1, 9)$, $E(2, 9)$ and $E(1, 2, 9)$, we get a reversal with $I = 16^+$ above $I = 0^+$. The respective values are 0.920, 3.332 and 4.947 MeV. We of course know that all even-even nuclei have $I = 0^+$ ground states. Indeed with the realistic CCGI interaction[2], the $I = 16^+$ state is at an excitation energy of 5.244 MeV (above the $I = 0^+$ ground state)[1].

What are the manifestations of J_{max} pairing or more generally of the proton-neutron interaction? This is most easily discussed by considering a system of 2 protons and 2 neutrons described by a wavefunction $\Sigma C(J_p J_n)[J_p J_n]$. This is actually an old story.

The values of $C(J, J)$ for various schematic interactions in the $g_{9/2}$ shell are here given as well as the $I = 16^+ - I = 0^+$ splitting (in square brackets).

Table 2: $C(J, J)$ for Various Interactions

Interaction	$C(J, J)$	I Splitting
E(0)	(0.8563, 0.1714, 0.2335, 0.2807, 0.3210)	[4.4000] J=0 pairing
E(9)	(.6104, 0.7518, 0.2328, 0.0233, 0.0005)	[-1.0589] J_{max} pairing
E(1)	(-0.4675, 0.3836, 0.0725, 0.7174, 0.3836)	[3.6494]
E(0,9)	(0.8013, 0.4814, 0.2514, 0.1718, 0.1833)	[2.1678]
E(1,9)	(0.5202, 0.7271, 0.3724, 0.0831, -0.2335)	[0.9025]
E(2,9)	(0.3701, 0.9602, 0.1077, -0.1585, -0.0722)	[3.3332]
E(1,2,9)	(0.3765, 0.8787, 0.2098, -0.0922, -0.1833)	[4.9467]
CCGI	(0.7725, 0.5289, 0.2915, 0.1704, 0.10210)	[5.2447]

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Appendices

Overlaps for even I and odd J_B ($T = 0$):

Table 3: $I = 2$

$E \backslash J_B$	9	7
1.0589	0.999959	0.000503096
3.0558	-0.000175206	1.00003
4.058	0	0
5.0588	0	0
5.0588	0	0
5.0588	0	0
5.0588	0	0
6.0588	0	0
6.0588	0	0
6.0588	0	0
8.0588	0	0
8.0588	0	0

Table 4: $I = 4$

$E \backslash J_B$	9	7	5
1.0588	1.00003	-0.000373778	0.00285081
3.028	-0.000954468	0.626255	0.786196
3.0715	-0.00105456	-0.779731	0.617937
4.0308	0	0	0
4.0605	0	0	0
5.0588	0	0	0
5.0588	0	0	0
5.0588	0	0	0
5.0588	0	0	0
6.0588	0	0	0
6.0588	0	0	0
6.0588	0	0	0
6.0588	0	0	0
8.0588	0	0	0
8.0588	0	0	0
8.0588	0	0	0

Table 5: $I = 6$

$E \backslash J_B$	9	7	5	3
1.0588	1.00003	0.00119549	-0.00305693	0.0110501
2.8527	-0.00220375	0.242714	0.805849	0.588035
3.0184	-0.00390295	0.715143	-0.523519	0.463442
3.1465	-0.00375281	-0.65544	-0.276701	0.662776
3.8532	0	0	0	0
4.0549	0	0	0	0
4.1357	0	0	0	0
5.0588	0	0	0	0
5.0588	0	0	0	0
5.0588	0	0	0	0
6.0588	0	0	0	0
6.0588	0	0	0	0
6.0588	0	0	0	0
6.0588	0	0	0	0
8.0588	0	0	0	0
8.0588	0	0	0	0
8.0588	0	0	0	0

Table 6: $I = 8$

$E \setminus J_B$	9	7	5	3	1
1.0571	0.999799	0.00249462	0.0103254	-0.0147855	0.0332888
2.0589	-0.0037983	0.0689463	0.460992	0.850713	0.562402
2.8436	-0.0155564	0.725503	0.149803	-0.399347	0.568632
3.0588	0	0	0	0	0
3.1677	-0.0112242	-0.633338	0.531456	-0.340893	0.34615
3.5299	0.00485083	0.260426	0.694712	-0.0172994	-0.489146
3.8645	0	0	0	0	0
4.0697	0	0	0	0	0
4.5287	0	0	0	0	0
5.0588	0	0	0	0	0
5.0588	0	0	0	0	0
6.0588	0	0	0	0	0
6.0588	0	0	0	0	0
8.0588	0	0	0	0	0
8.0588	0	0	0	0	0

Table 7: $I = 10$

$E \setminus J_B$	9	7	5	3	1
1.0464	0.997851	0.0321553	0.00357842	0.0887311	-0.11316
2.0626	-0.0423012	0.396662	0.860318	-0.141699	-0.721587
2.8207	-0.0433641	0.674428	-0.505476	0.761326	-0.440961
3.0601	0	0	0	0	0
3.539	-0.0259723	-0.621868	-0.0658534	0.626445	-0.521579
4.0217	0	0	0	0	0
4.5109	0	0	0	0	0
5.0588	0	0	0	0	0
6.0588	0	0	0	0	0
8.0588	0	0	0	0	0

Overlaps for even I and even J_B ($T = 1$):

Table 8: $I = 2$

$E \setminus J_B$	8
1.0589	0
3.0558	0
4.058	0.999981
5.0588	0
5.0588	0
5.0588	0
5.0588	0
6.0588	0
6.0588	0
6.0588	0
8.0588	0
8.0588	0

Table 9: $I = 4$

$E \backslash J_B$	8	6
1.0588	0	0
3.028	0	0
3.0715	0	0
4.0308	0.160936	0.987333
4.0605	-0.986952	0.158686
5.0588	0	0
5.0588	0	0
5.0588	0	0
5.0588	0	0
6.0588	0	0
6.0588	0	0
6.0588	0	0
6.0588	0	0
8.0588	0	0
8.0588	0	0
8.0588	0	0

Table 10: $I = 6$

$E \backslash J_B$	8	6	4
1.0588	0	0	0
2.8527	0	0	0
3.0184	0	0	0
3.1465	0	0	0
3.8532	0.0575001	0.530782	0.875848
4.0549	0.979765	-0.19661	0.0512894
4.1357	-0.191929	-0.824416	0.479818
5.0588	0	0	0
5.0588	0	0	0
5.0588	0	0	0
6.0588	0	0	0
6.0588	0	0	0
6.0588	0	0	0
6.0588	0	0	0
8.0588	0	0	0
8.0588	0	0	0
8.0588	0	0	0

Table 11: $I = 8$

$E \backslash J_B$	8	6	4	2
1.0571	0	0	0	0
2.0589	0	0	0	0
2.8436	0	0	0	0
3.0588	0.0163788	0.201778	0.702887	0.854136
3.1677	0	0	0	0
3.5299	0	0	0	0
3.8645	0.242862	0.800773	-0.534145	0.252455
4.0697	-0.967177	0.234518	-0.068364	0.022665
4.5287	-0.0726641	-0.512835	-0.464627	0.454166
5.0588	0	0	0	0
5.0588	0	0	0	0
6.0588	0	0	0	0
6.0588	0	0	0	0
8.0588	0	0	0	0
8.0588	0	0	0	0

Table 12: $I = 10$

$E \backslash J_B$	8	6	4	2
1.0464	0	0	0	0
2.0626	0	0	0	0
2.8207	0	0	0	0
3.0601	0.126944	0.730409	0.788843	-0.941105
3.539	0	0	0	0
4.0217	0.960518	-0.279975	0.119751	-0.0470487
4.5109	0.247442	0.623127	-0.602827	0.334915
5.0588	0	0	0	0
6.0588	0	0	0	0
8.0588	0	0	0	0

Overlaps for odd I and odd J_B ($T = 1$):

Table 13: $I = 3$

$E \backslash J_B$	8	6
3.0658	0	0
4.0556	0.886512	-0.465285
4.0694	-0.462764	-0.885217
5.0588	0	0
6.0588	0	0
6.0588	0	0
6.0588	0	0
6.0588	0	0
6.0588	0	0
8.0588	0	0

Table 14: $I = 5$

$E \backslash J_B$	8	6	4
3.0394	0	0	0
3.1403	0	0	0
4.0228	0.521481	0.528141	-0.690876
4.0696	-0.838204	0.461911	-0.288948
4.1421	-0.159705	-0.712513	-0.662573
5.0588	0	0	0
5.0588	0	0	0
6.0588	0	0	0
6.0588	0	0	0
6.0588	0	0	0
6.0588	0	0	0
6.0588	0	0	0
8.0588	0	0	0

Table 15: $I = 7$

$E \backslash J_B$	8	6	4	2
2.8558	0	0	0	0
3.1128	0	0	0	0
3.5294	0	0	0	0
3.8495	0.242949	0.74858	0.120268	-0.685731
4.0244	0.800366	-0.487776	0.314524	-0.224686
4.1601	-0.547016	-0.360402	0.533307	-0.535697
4.5295	-0.0329213	-0.267937	-0.775985	-0.438508
5.0588	0	0	0	0
6.0588	0	0	0	0
6.0588	0	0	0	0
6.0588	0	0	0	0
6.0588	0	0	0	0
8.0588	0	0	0	0

Table 16: $I = 9$

$E \backslash J_B$	8	6	4	2
2.0588	0	0	0	0
2.907	0	0	0	0
3.0586	-0.063442	-0.474189	-0.854903	-0.000975866
3.5251	0	0	0	0
3.8098	0.641427	0.30965	-0.434754	0.753832
4.1659	-0.727682	0.514185	-0.282222	0.341678
4.5362	-0.234441	-0.644132	0.021184	0.561164
6.0588	0	0	0	0
6.0588	0	0	0	0
8.0588	0	0	0	0

Overlaps for odd I and odd J_B ($T = 0$):

Table 17: $I = 3$

$E \setminus J_B$	9
3.0658	-1.00006
4.0556	0
4.0694	0
5.0588	0
6.0588	0
6.0588	0
6.0588	0
6.0588	0
6.0588	0
8.0588	0

Table 18: $I = 5$

$E \setminus J_B$	9	7
3.0394	-0.933845	0.37385
3.1403	0.357561	0.927537
4.0228	0	0
4.0696	0	0
4.1421	0	0
5.0588	0	0
5.0588	0	0
6.0588	0	0
6.0588	0	0
6.0588	0	0
6.0588	0	0
6.0588	0	0
8.0588	0	0

Table 19: $I = 7$

$E \setminus J_B$	9	7	5
2.8558	-0.472944	-0.73643	0.636574
3.1128	0.874802	-0.431482	0.189249
3.5294	0.105405	0.521038	0.747663
3.8495	0	0	0
4.0244	0	0	0
4.1601	0	0	0
4.5295	0	0	0
5.0588	0	0	0
6.0588	0	0	0
6.0588	0	0	0
6.0588	0	0	0
6.0588	0	0	0
8.0588	0	0	0

Table 20: $I = 9$

$E \backslash J_B$	9	7	5	3
2.0588	-0.194963	-0.741037	-0.769821	0.929465
2.907	-0.877203	0.463008	-0.267855	0.124265
3.0586	0	0	0	0
3.5251	0.43861	0.486313	-0.579331	0.347367
3.8098	0	0	0	0
4.1659	0	0	0	0
4.5362	0	0	0	0
6.0588	0	0	0	0
6.0588	0	0	0	0
8.0588	0	0	0	0